Counting paths in lattices to obtain symmetric polynomial identities

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Theorem (Binomial determinant duality, Gessel–Viennot, 85)

Let $n \geqslant 0$, let A and B be subsets of $\{0, 1, \ldots, n\}$ of equal size, and let A^{c} and B^{c} be their complements in $\{0, 1, \ldots, n\}$.

$$\det\left(\binom{b}{a}\right)_{a \in A, b \in B} = \det\left(\binom{a'}{b'}\right)_{a' \in A^{c}, b' \in B^{c}}$$

Symmetric polynomials

Variables $x_1, x_2, x_3, \ldots$
Symmetric: interchanging x_i and x_j doesn't change the polynomial
Elementary symmetric polynomials:

$$e_d(x_1, x_2, \ldots, x_r) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant r} x_{i_1} x_{i_2} \cdots x_{i_d}$$

Complete homogeneous symmetric polynomials:

$$h_d(x_1, x_2, \ldots, x_r) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_d \leqslant r} x_{i_1} x_{i_2} \cdots x_{i_d}$$

Examples:

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$
$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3$$

Symmetric polynomials

$$e_{a'-b'}(\underbrace{1,1,\ldots,1}_{a'}) = \binom{a'}{a'-b'} = \binom{a'}{b'}$$

$$h_{b-a}(\underbrace{1,1,\ldots,1}_{a+1}) = \left(\!\!\binom{a+1}{b-a}\!\!\right) = \binom{a+1+b-a-1}{b-a} = \binom{b}{a}$$

Theorem (McD, 20)

Let $n \geqslant 0$, let A and B be subsets of $\{0,1,\ldots,n\}$ of equal size, and let A^{c} and B^{c} be their complements in $\{0,1,\ldots,n\}$. Then

$$\det\Big(h_{b-a}(x_1,x_2,\ldots,x_{a+1})\Big)_{a\in A, b\in B}$$

$$=$$

$$\det\Big(e_{a'-b'}(x_1,x_2,\ldots,x_{a'})\Big)_{a'\in A^{\mathrm{c}}, b'\in B^{\mathrm{c}}}$$

Theorem (Aitken, 31)

Let $n \geqslant 0$, let A and B be subsets of $\{0, 1, \ldots, n\}$ of equal size, and let A^{c} and B^{c} be their complements in $\{0, 1, \ldots, n\}$. Let $m \geqslant 1$. Then

$$\det\Big(h_{b-a}(x_1, x_2, \ldots, x_m)\Big)_{a\in A, b\in B}$$

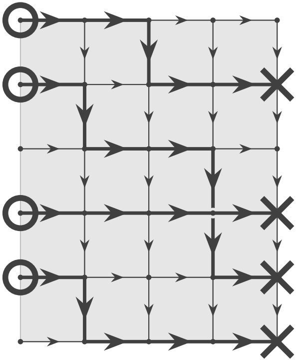
$$=$$

$$\det\Big(e_{a'-b'}(x_1, x_2, \ldots, x_m)\Big)_{a'\in A^{\mathsf{c}}, b'\in B^{\mathsf{c}}}$$

Lattices and paths

Lattice: directed, acyclic graph with sources and sinks
Path: sequence of edges from a source to a sink

Example:

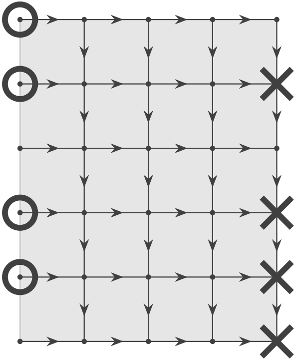


Lindström–Gessel–Viennot lemma

Let M be the matrix with

$$M_{i,j} = \text{ number of paths from } i\text{th source to } j\text{th sink}$$

Example:



$$M_{2,3} = \left(\!\!\binom{4}{3}\!\!\right) = \binom{6}{3} = 20$$

Lindström–Gessel–Viennot lemma

Lemma (Lindström–Gessel–Viennot)

Suppose the lattice is nonpermutable. Then

$$\det M \;=\; \text{number of non-intersecting tuples of paths}$$

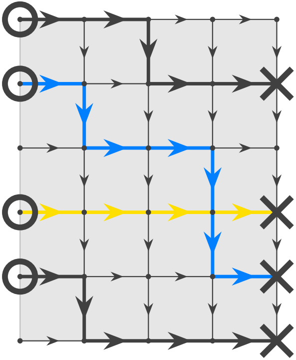
Use the Leibniz formula

$$\det M = \sum_{\sigma \in S_l} \operatorname{sgn}(\sigma) M_{1,\sigma(1)} \cdots M_{l,\sigma(l)}$$

Notice that $M_{1,\sigma(1)} \cdots M_{l,\sigma(l)}$ is the total number of tuples of paths such that the ith source goes to the $\sigma(i)$th sink

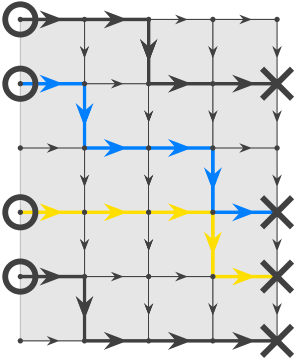
Lindström–Gessel–Viennot lemma

Intersecting paths cancel out:



$\sigma = (2\ 3)$

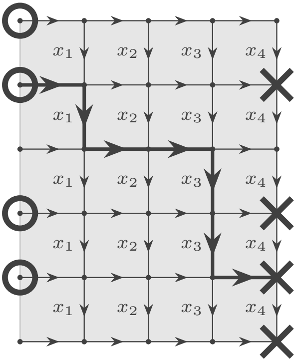
$\text{sgn}(\sigma) = -1$



$\sigma = \text{id}$

$\text{sgn}(\sigma) = 1$

Weighted paths



weight of path $= x_1 x_3^2$

$$M_{2,3} = h_3(x_1, x_2, x_3, x_4)$$

Strategy

Aim: given $n \geqslant 0$, $m \geqslant 1$ and $A, B \subseteq \{0, 1, \ldots, n\}$, want to show

$$\det\Big(h_{b-a}(x_1, \ldots, x_m)\Big)_{a \in A, b \in B} = \det\Big(e_{a'-b'}(x_1, \ldots, x_m)\Big)_{a' \in A^c, b' \in B^c}$$

Strategy:

- Define a lattice with sources and sinks indexed by A and B with

$$\begin{matrix}\text{(weighted) number of paths} \\ \text{from source } a \text{ to sink } b\end{matrix} = h_{b-a}(x_1, \ldots, x_m)$$

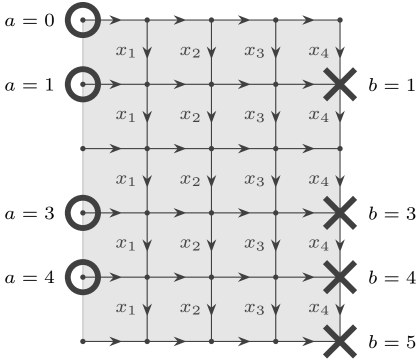
  for all $a \in A$, $b \in B$
- Define a lattice with sources and sinks indexed by B^c and A^c with

$$\begin{matrix}\text{(weighted) number of paths} \\ \text{from source } b' \text{ to sink } a'\end{matrix} = e_{a'-b'}(x_1, \ldots, x_m)$$

  for all $a' \in A^c$, $b \in B^c$
- Construct a (weight-preserving) bijection between non-intersecting tuples of paths on these lattices

First lattice

$$n = 5,\ m = 4,\ A = \{0, 1, 3, 4\},\ B = \{1, 3, 4, 5\}$$

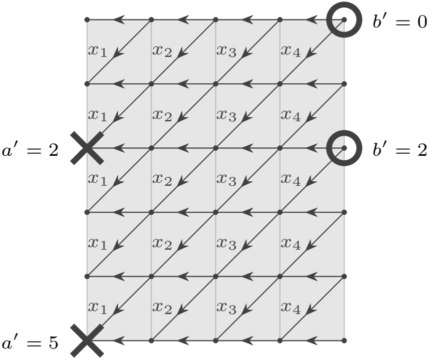


$$M_{a,b} = h_{b-a}(x_1, \dots, x_m)$$

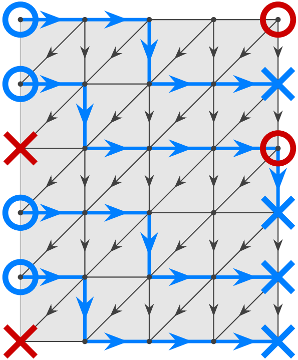
Second lattice

$$n = 5,\ m = 4,\ A = \{0, 1, 3, 4\},\ B = \{1, 3, 4, 5\}$$

$$M_{b',a'} = e_{a'-b'}(x_1, \ldots, x_m)$$



Bijection between non-intersecting tuples of paths



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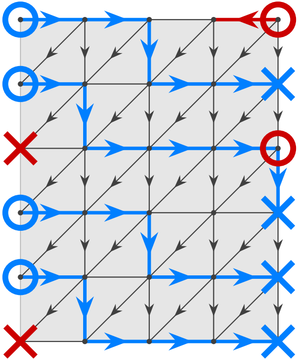

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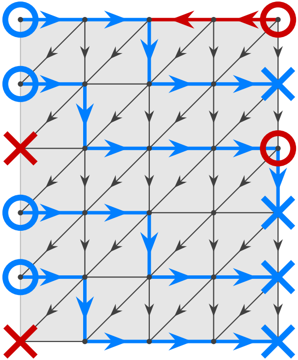

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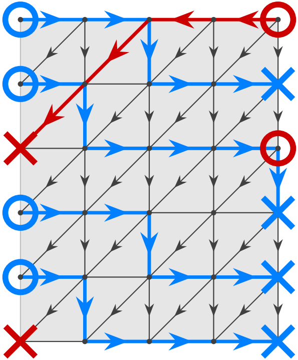

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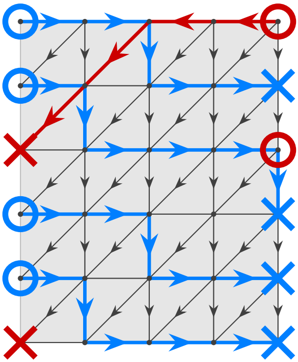
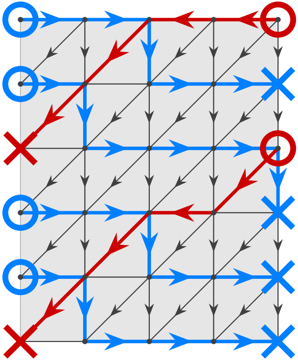
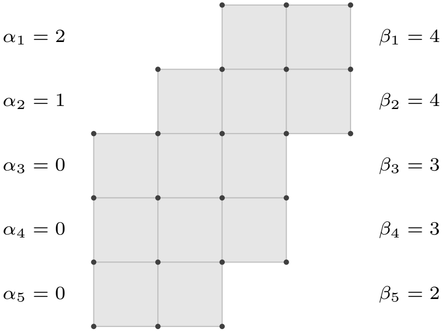


Bijection between non-intersecting tuples of paths



Lattices of more general shape

Sequences α and β control number of boxes in each row
(non-increasing, n terms, with $\alpha_i \leqslant \beta_i$ for each i)



Condition on lattice for result to hold?
Sufficient that α and β decrease by at most 1 at each step

Lattices of more general shape

Theorem (McD, 20)

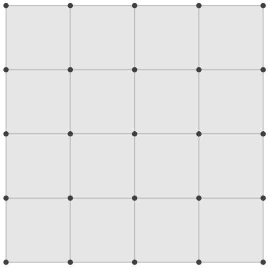
Let $n \geqslant 0$, let A and B be subsets of $\{0, 1, \ldots, n\}$ of equal size, and let A^c and B^c be their complements in $\{0, 1, \ldots, n\}$. Let α and β be partitions with n parts (with parts equal to 0 permitted) such that $\alpha_i \leqslant \beta_i$ for all $i \in \{0, 1, \ldots, n\}$.
Suppose each part of α and β is at most 1 less than the preceding part.
Then

$$\det\Big(h_{b-a}(x_{\alpha_{a+1}+1}, x_{\alpha_{a+1}+2}, \ldots, x_{\beta_b})\Big)_{a \in A, b \in B}$$

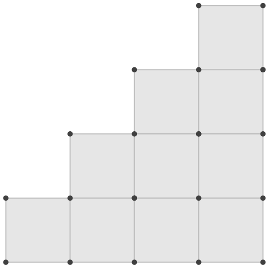
$$=$$

$$\det\Big(e_{a'-b'}(x_{\alpha_{a'}+1}, x_{\alpha_{a'}+2}, \ldots, x_{\beta_{b'+1}})\Big)_{a' \in A^c, b' \in B^c}$$

Lattices of more general shape



⇝ Aitken symmetric function duality theorem



⇝ Gessel–Viennot binomial duality theorem

Application: plethysm in representation theory

Let k be a field
Let $G = \mathrm{SL}_2(k)$ be the special linear group of 2×2 matrices over k with determinant 1
Let E be the (2-dimensional) natural representation of G

The *Schur functor* (corresponding to a partition λ) on the category of representations of G is denoted ∇^λ

Question (problem of *plethysm*): what is $\nabla^\mu \nabla^\lambda E$?

Symmetric powers (Sym^r) and exterior powers ($\bigwedge^r$) are examples of Schur functors

Classical result: $\bigwedge^r \mathrm{Sym}^m E \cong \bigwedge^{m+1-r} \mathrm{Sym}^m E$

Application: plethysm in representation theory

$\mathrm{Sym}^m E$ is $(m+1)$-dimensional, say with basis $\{v_0, v_1, \ldots, v_m\}$

Then $\bigwedge^r \mathrm{Sym}^m E$ is $\binom{m+1}{r}$-dimensional with basis indexed by r-subsets of $\{0, 1, \ldots, m\}$,
and $\bigwedge^{m+1-r} \mathrm{Sym}^m E$ is $\binom{m+1}{m+1-r}$-dimensional with basis indexed by $(m+1-r)$-subsets of $\{0, 1, \ldots, m\}$.

Proposed map: $A \mapsto m - A^{\mathrm{c}}$

Consider action of a transvection $g = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ on each side

In $\bigwedge^r \mathrm{Sym}^m E$: coefficient of A in gB is $\det\Big(\binom{b}{a}\Big)_{a \in A, b \in B}$

In $\bigwedge^{m+1-r} (\mathrm{Sym}^m E)^{\circ}$:
coefficient of $m - A^{\mathrm{c}}$ in $g(m - B^{\mathrm{c}})$ is $\det\Big(\binom{a'}{b'}\Big)_{a' \in A^{\mathrm{c}}, b' \in B^{\mathrm{c}}}$