Determination of characters by their values on p'-classes

Eoghan McDowell

Okinawa Institute of Science and Technology

May 25, 2022

p'-classes

Let G be a group. Let p be a prime.

A conjugacy class $\mathcal{C}$ of G is called a p'-class if p does not divide the order of the elements of $\mathcal{C}$.

Question: can an ordinary irreducible character of G be uniquely determined by its values on the p'-classes?

Rows of the decomposition matrix

Let χ and ψ be ordinary irreducible characters of G.

The following are equivalent:

- χ and ψ agree on the p'-classes of G;
- the rows of the p-modular decomposition matrix labelled by χ and ψ are equal;
- the p-modular reductions of the representations affording χ and ψ have the same multiset of composition factors.

D	B	$=$	X
decomposition matrix	Brauer character table		ordinary character table on p'-classes

Example: cyclic group

Let $G = C_p = \langle x \rangle$ be a cyclic group of order p.
Let ζ be a primitive pth root of unity in $\mathbb{C}$.

1	x	x^2	$\cdots$	x^{p-1}
1	1	1	$\cdots$	1
1	ζ	ζ^2	$\cdots$	ζ^{p-1}
1	ζ^2	ζ^4	$\cdots$	$\zeta^{2(p-1)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
1	ζ^{p-1}	$\zeta^{2(p-1)}$	$\cdots$	$\zeta^{(p-1)^2}$

$$D = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Example: symmetric group S_5

Let $G = S_5$ be the symmetric group on five symbols. Let $p = 5$.

(1^5)	$(2,1^3)$	$(2^2,1)$	$(3,1^2)$	$(3,2)$	$(4,1)$	(5)
1	1	1	1	1	1	1
1	-1	1	1	-1	-1	1
4	2	0	1	-1	0	-1
4	-2	0	1	1	-1	-1
5	1	1	-1	1	1	0
5	-1	1	-1	-1	-1	0
6	0	-2	0	0	0	1

$$D = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ 1 & & 1 & & & \\ & 1 & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & 1 & 1 & & \end{pmatrix}$$

Example: alternating group A_5

Let $G = A_5$ be the alternating group on five symbols. Let $p = 5$.

(1^5)	$(2^2, 1)$	$(3, 1^2)$	$(5)^+$	$(5)^-$
1	1	1	1	1
3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
4	0	1	-1	-1
5	1	-1	0	0

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$$

Results for symmetric and alternating groups

Theorem (Wildon 08)

Let $p \neq 2$.
The characters of S_n are uniquely determined by their values on p'-classes.

Theorem (McD. 22)

Let $p \neq 3$.
The characters of A_n are uniquely determined by their values on p'-classes, except for the split pair of characters labelled by a partition with a diagonal hook length divisible by p.

Theorem (McD. 22)

Let $p \neq 3$ and let $G = \widetilde{A}_n$ or $G = \widetilde{S}_n$.
The negative characters of G are uniquely determined by their values on p'-classes, except for the pairs of associate or conjugate characters labelled by a partition with a part divisible by p.

Characters of symmetric and alternating groups

Ordinary irreducible characters of S_n are labelled by partitions of n. Write χ^λ for the character corresponding to a partition λ.

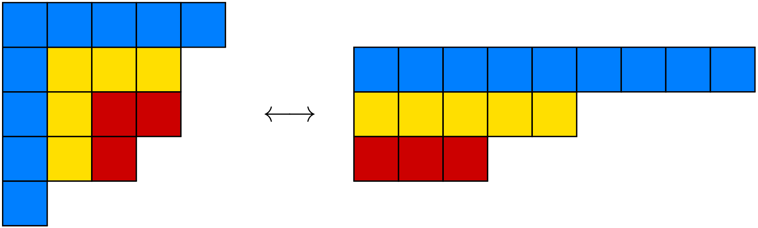
On restriction to A_n:

- If $\lambda \neq \lambda'$, then $\chi^\lambda\downarrow_{A_n} = \chi^{\lambda'}\downarrow_{A_n}$ is irreducible.
- If $\lambda = \lambda'$, then $\chi^\lambda\downarrow_{A_n}$ splits into two irreducible characters of A_n denoted χ^{λ^+} and χ^{λ^-}.

Conjugacy classes of symmetric and alternating groups

Conjugacy classes of S_n are determined by cycle type, and the elements have order the gcd of the cycle type.
A conjugacy class of S_n splits in A_n if and only if it has cycle type consisting of distinct odd integers.

There is a bijection between self-conjugate partitions and partitions consisting of distinct odd integers:



For λ self-conjugate, the characters χ^{λ^+} and χ^{λ^-} of A_n differ only on the split class of cycle type the diagonal hook lengths of λ.

Characters of A_n which agree on p'-classes

For λ self-conjugate, the characters χ^{λ^+} and χ^{λ^-} of A_n differ only on the split class of cycle type the diagonal hook lengths of λ.

Thus if λ is self-conjugate and has a diagonal hook length divisible by p, the characters χ^{λ^+} and χ^{λ^-} of A_n differ only on a class of elements of order divisible by p.
That is, χ^{λ^+} and χ^{λ^-} agree on p'-classes.

Theorem (McD. 22)

Let $p \neq 3$.
The characters of A_n are uniquely determined by their values on p'-classes, ***except for the split pair of characters labelled by a partition with a diagonal hook length divisible by p.***

Number of p'-classes

Fix p and let n grow.

[Hardy–Ramanujan, 18] Number of partitions of n

$$\sim An^{-1} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$$

[Hagis, 71] Number of partitions of n with no part divisible by p

$$\sim Bn^{-\frac{3}{4}} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{\frac{p-1}{p}}\sqrt{n}\right)$$

So proportion of partitions of n with no part divisible by p

$$\sim Cn^{\frac{1}{4}} \exp\left(-\gamma\sqrt{n}\right)$$

for some $\gamma > 0$

Ingredient 1: central characters

A *central character* of a group G is defined to be an algebra homomorphism $Z(\mathbb{Q}G) \to \mathbb{C}$.

For every ordinary irreducible character χ of G, there is a corresponding central character ω_χ: if V is the representation affording χ and $z \in Z(\mathbb{Q}G)$, then by Schur's Lemma z acts on V by a scalar; define $\omega_\chi(z)$ to be that scalar.

The centre $Z(\mathbb{Q}G)$ has a linear basis the set of *conjugacy class sums*.
If $\mathcal{C}$ is a conjugacy class of G, let $s_\mathcal{C} \in Z(\mathbb{Q}G)$ denote the sum of the elements of $\mathcal{C}$.

If χ is an ordinary irreducible character of G and $\mathcal{C}$ is a conjugacy class of G, then the corresponding central character has values given by

$$\omega_\chi(s_\mathcal{C}) = |\mathcal{C}| \frac{\chi(\mathcal{C})}{\chi(1)}.$$

Ingredient 1: central characters

If χ is an ordinary irreducible character of G and $\mathcal{C}$ is a conjugacy class of G, then the corresponding central character has values given by

$$\omega_\chi(s_\mathcal{C}) = |\mathcal{C}| \frac{\chi(\mathcal{C})}{\chi(1)}.$$

Lemma

Let $\mathcal{U}$ be a collection of conjugacy classes of G (with $\{1\} \in \mathcal{U}$). Suppose $\{ s_\mathcal{C} \mid \mathcal{C} \in \mathcal{U} \}$ generates $Z(\mathbb{Q}G)$ as an algebra. Then irreducible characters of G are uniquely determined by their values on the classes in $\mathcal{U}$.

Proof.

Suppose χ, ψ are irreducible characters of G which agree on $\mathcal{U}$.
Then ω_χ and ω_ψ agree on $\{ s_\mathcal{C} \mid \mathcal{C} \in \mathcal{U} \}$.
Then $\omega_\chi = \omega_\psi$, and hence $\chi = \psi$. □

Ingredient 2: induction on support of a partition

Define the *support* of a partition to be the number of non-fixed points of a permutation of cycle type given by the partition.

To illustrate how this is useful, we will prove:

Proposition

Ordinary irreducible characters of S_n are uniquely determined by their values on cycles.

Let $s_\lambda \in Z(\mathbb{Q}S_n)$ denote the sum of the conjugacy class of cycle type λ.

It suffices to show that $\{ s_{(i)} \mid 1 \leqslant i \leqslant n \}$, the set of class sums of cycles, generate $Z(\mathbb{Q}S_n)$.

We will show that s_λ is in the algebra generated by $\{ s_{(i)} \mid 1 \leqslant i \leqslant n \}$, by induction on the support of λ.

Ingredient 2: induction on support of a partition

Claim

The class sum s_λ *is in the algebra generated by* $\{\, s_{(i)} \mid 1 \leqslant i \leqslant n \,\}$.

Proof.

If $\lambda = (i)$ has only one part, the claim is trivial.

Otherwise, let $\bar{\lambda}$ denote the partition λ with first part removed.

By the inductive hypothesis, $s_{(\lambda_1)}$ and $s_{\bar{\lambda}}$ are in the algebra, and hence so is the product $s_{(\lambda_1)} s_{\bar{\lambda}}$.

The class sum s_λ is the only class sum appearing in this product labelled by a partition with support equal to that of λ. $\square$

Ingredient 3: conjugacy of characters

We already found some irreducible characters of A_n which are not uniquely determined by their values on p'-classes – so it is not true that the p'-class sums generate $Z(\mathbb{Q}A_n)$.

But it can be shown that the p'-class sums generate enough of $Z(\mathbb{Q}A_n)$ to determine characters up to S_n-conjugacy.

Then S_n-conjugate characters of A_n are precisely the split characters (labelled by self-conjugate partitions), and it is clear when these can be determined by their values on p'-classes.

$p = 3$

Theorem

Let $n \geqslant 3$ and let ν be a self-conjugate partition of $n - 3$.

(i) *Suppose ν is 3-core.*
Let λ be the partition of n obtained by adding a 3-hook to the first row of ν, and let μ be the self-conjugate partition of n obtained by adding a diagonal 3-hook to ν.
Then $\chi^{\lambda}\downarrow_{A_n}$, χ^{μ^+} and χ^{μ^-} agree on the $3'$-classes.

(ii) *Suppose ν has a unique 3-hook (necessarily on the diagonal).*
Let λ, λ', μ and μ' be the four partitions of n which can be obtained from ν by adding a 3-hook.
Then $\chi^{\lambda}\downarrow_{A_n}$ and $\chi^{\mu}\downarrow_{A_n}$ agree on the $3'$-classes.

Example: alternating group A_4, $p = 3$

Let $G = A_4$ be the alternating group on four symbols. Let $p = 3$. Let ω be a primitive cube root of unity in $\mathbb{C}$.

(1^5)	$(2,2)$	$(3,1)^+$	$(3,1)^-$
1	1	1	1
1	1	ω	ω^2
1	1	ω^2	ω
3	-1	0	0

$$D = \begin{pmatrix} 1 & \\ 1 & \\ 1 & \\ & 1 \end{pmatrix}$$

$p = 3$ (proof of theorem)

Follows from:

Theorem (James)

Let ν be a partition of $n - r$.
Consider the sum $\sum_\lambda (-1)^i \chi^\lambda$ over all partitions λ obtained from ν by adding an r-hook (where i is the leg length of the added hook).
This sum vanishes on all classes except those containing an r-cycle.

In case (i) of the $p = 3$ theorem: on $3'$-classes,

$$\chi^\lambda + \chi^{\lambda'} - \chi^\mu = 0.$$

That is, on $3'$-classes,

$$2\chi^\lambda\downarrow_{A_n} = \chi^{\mu^+} + \chi^{\mu^-}.$$

Thank you