Plethysm, the inverse Schur functor, and other connections between representation theory and combinatorics

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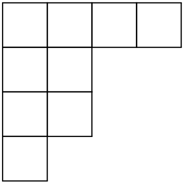
Introduction: tableaux and Specht modules

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$



Young diagram $[\lambda]$
of $\lambda = (4, 2, 2, 1)$

1	2	4	6
3	7		
5	9		
8			

Row tabloid of shape λ

The symmetric group S_n acts on tableaux and tabloids

Introduction: tableaux and Specht modules

A *polytabloid* is the linear combination of row tabloids defined by

$$e(t) = \sum_{\sigma \in \mathrm{CPP}(\lambda)} (\operatorname{sgn} \sigma)[t \cdot \sigma]$$

The space of polytabloids is the *Specht module* S^λ

In characteristic zero, $\{ S^\lambda \mid \lambda \vdash n \}$ is a complete set of simple modules

In characteristic $p > 0$, $\{ D^\lambda \mid \lambda \vdash n, \lambda \text{ is } p\text{-regular} \}$ is a complete set of simple modules, where $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$

Schur endofunctors

Now consider tableaux whose entries are basis vectors for an arbitrary representation (and no longer require tableaux to be bijections)

$$\text{representation } V \rightsquigarrow \text{ space of polytabloids } \nabla^\lambda(V)$$

∇^λ is an endofunctor on the category of representations of a given group

Choosing $V = E$ the natural representation of the general linear group $\mathrm{GL}_d(K)$, the module $\nabla^\lambda(E)$ is a *dual Weyl module*

In characteristic zero, $\{\, \nabla^\lambda(E) \mid \lambda \vdash n \,\}$ is a complete set of simple modules

In characteristic $p > 0$, $\{\, \mathrm{soc}\, \nabla^\lambda(E) \mid \lambda \vdash n \,\}$ is a complete set of simple modules

Schur endofunctors

Example:
if $\lambda = (n)$ is a single row, then $\nabla^\lambda = \mathrm{Sym}^n$ is the symmetric power

Example:
if $\lambda = (1^n)$ is a single column, then $\nabla^\lambda = \bigwedge^n$ is the exterior power

Plethysm

Plethysm is the composition of Schur endofunctors

Goal:
Describe $\nabla^\mu \nabla^\lambda(E)$, for E the natural representation of $\mathrm{GL}_d(K)$

Connection with combinatorics:
The character of $\nabla^\lambda(E)$ is the *Schur polynomial* s_λ
The character of $\nabla^\mu \nabla^\lambda(E)$ is the *plethysm* product $s_\mu \circ s_\lambda$

Plethystic isomorphisms for $\mathrm{SL}_2(K)$

Classical ($K = \mathbb{C}$) results [King, 85]:

- $\mathrm{Sym}^a \mathrm{Sym}^b E \cong \bigwedge^a \mathrm{Sym}^{a+b-1} E$
- $\nabla^\lambda \mathrm{Sym}^{a-1} E \cong \nabla^{\lambda^\circ} \mathrm{Sym}^{a-1} E$
- $\nabla^{(a+1,1^b)} \mathrm{Sym}^{m+a} E \cong \nabla^{(b+1,1^a)} \mathrm{Sym}^{m+b} E$

Here the *box complement* λ° is obtained by taking the complement of the set of boxes of λ inside a given rectangle

Theorem (McD.–Wildon)

In arbitrary characteristic, there are isomorphisms

- $\mathrm{Sym}_a \mathrm{Sym}^b E \cong \bigwedge^a \mathrm{Sym}^{a+b-1} E$
- $\nabla^\lambda \mathrm{Sym}^{a-1} E \cong \nabla^{\lambda^\circ} \mathrm{Sym}_{a-1} E$

but no combination of dualities generalises the isomorphism $\nabla^{(a+1,1^b)} \mathrm{Sym}^{m+a} E \cong \nabla^{(b+1,1^a)} \mathrm{Sym}^{m+b} E$

The Schur functor and its inverse

Schur functor:

$\mathcal{F}\colon \{\text{repns of } \mathrm{GL}_d(K) \text{ (polynomial of degree } n)\} \to \{\text{repns of } S_n\}$

Inverse Schur functor (right-inverse to $\mathcal{F}$):

$\mathcal{G}\colon \{\text{repns of } S_n\} \to \{\text{repns of } \mathrm{GL}_d(K) \text{ (polynomial of degree } n)\}$

Known:

- $\mathcal{F}(\nabla^\lambda(E)) \cong S^\lambda$
- $\mathcal{G}(S^\lambda) \cong \nabla^\lambda(E)$ when $p \geqslant 5$ [Kleshchev–Nakano, 01]

Question:

- Does $\mathcal{G}(S^\lambda) \cong \nabla^\lambda(E)$ hold when $p = 2$ or $p = 3$?

Answer:

- for $p = 3$: yes
- for $p = 2$: depends on λ

The Specht module under the inverse Schur functor

Theorem (McD. 21)

If char $K \neq 2$, *then there is an isomorphism* $\mathcal{G}(S^\lambda) \cong \nabla^\lambda(E)$

Theorem (McD. 21)

Suppose char $K = 2$ *and* $d \geqslant n - 2$.
There is an isomorphism $\mathcal{G}(S^\lambda) \cong \nabla^\lambda E$ *if and only if* λ *is* 2*-regular, or* $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ *and* λ *minus its first part is* 2*-regular.*

Intermediate step:
The isomorphism $\mathcal{G}(S^\lambda) \cong \nabla^\lambda(E)$ holds if and only if every column tabloid with a repeated column entry can be written as a linear combination of (skew) Garnir relations

Thank you